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Stability of the electroweak ground state in the Standard Model and its extensions



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ABSTRACT

We review the formalism by which the tunnelling probability of an unstable ground state can be computed in quantum field theory, with special reference to the Standard Model of electroweak interactions. We describe in some detail the approximations implicitly adopted in such calculation. Particular attention is devoted to the role of scale invariance, and to the different implications of scale-invariance violations due to quantum effects and possible new degrees of freedom. We show that new interactions characterized by a new energy scale, close to the Planck mass, do not invalidate the main conclusions about the stability of the Standard Model ground state derived in absence of such terms.

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1. Introduction

In recent years there has been considerable interest in the problem of the stability of the Standard Model (SM) ground state. Due to the sizable negative contribution to the β function of the Higgs self-coupling induced by top-quark loops, the usual electroweak vacuum $|0\rangle$, characterized by $\langle 0|H|0\rangle = v \approx 246$ GeV, may not be the absolute minimum of the scalar potential. In this case the true minimum of the theory is located at much larger energy scales and, in the absence of New Physics (NP) modifying the effective Higgs potential, the electroweak vacuum is unstable. The parameter space of the SM (with particular reference to the top-quark mass, m_t , and the Higgs-boson mass, m_h , which are the most relevant parameters) is thus naturally divided into three regions: stability, instability, and metastability. The stability region is the one where the electroweak vacuum is the absolute minimum of the potential. The instability and metastability regions are those where a new deeper minimum exists, with the metastability region being characterized by a lifetime of the unstable electroweak vacuum larger than the age of the Universe. More precisely, the instability/metastability boundary is determined by the decay prob-

ability of the electroweak vacuum under quantum tunnelling, that sets a model-independent upper bound on the lifetime of the unstable vacuum irrespective of the thermal history of the Universe.

A precise determination of the boundaries of these three regions has recently been presented in Refs. [1,2] (see also [3–5]). As a result of these recent analyses, the present experimental values of m_h and m_t lie in the metastability region of the SM parameter space. This finding holds only within the SM; however, it has an important consequence for beyond-the-SM searches: it implies that there is no need to invoke the presence of NP in order to stabilize the SM electroweak vacuum.

The validity of the analysis in Refs. [1,2] and the corresponding conclusions has been questioned in a series of recent papers [6–9]. There, it has been shown that non-standard physics modifying the shape of the Higgs potential at energy scales of the order of the Planck mass can sizeably affect the tunnelling rate of the electroweak vacuum. This observation is correct. However, as we discuss in the following, it does not invalidate the interest and the main conclusions of the analyses based on the SM potential.

In this paper we present a critical re-analysis of the problem of the SM vacuum stability. Our purpose is to clarify the assumptions and approximations employed in the evaluation of the SM tunnelling rate, with particular attention to those that have been often overlooked or implicitly adopted in the existing literature.

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2. The tunnelling rate within the SM

We begin by reviewing the standard formalism, originally worked out by Coleman and Callan [10,11], which allows one to compute the probability per unit time (or, equivalently, the lifetime) of a false ground state to a true ground state in quantum field theory. In the semiclassical approximation, the decay probability per unit time of the electroweak ground state is given by [12]

$$\Gamma \approx \frac{\tau_U^3}{R^4} e^{-S[h]}, \quad (1)$$

where τ_U is the age of the Universe,

$$S[h] = \int d^4x \left[\frac{1}{2} \partial_\mu h \partial_\mu h + V(h) \right] \quad (2)$$

is the euclidean action of the theory, computed for a specific solution h of the euclidean field equation for the scalar field which is usually called the *bounce*, and R is a dimensional factor associated with the size of the bounce. The bounce field configuration is such that it is equal to the false vacuum configuration $h = v$ at infinite euclidean time τ , and completes barrier penetration at $\tau = 0$.

A known result, conjectured by Coleman and subsequently proved by Coleman himself, Glaser and Martin [13], guarantees that the bounce solution of minimum action is invariant under four-dimensional rotations in euclidean spacetime, that is

$$h = h(r); \quad r^2 = |\vec{x}|^2 + \tau^2. \quad (3)$$

Hence,

$$\lim_{r \rightarrow \infty} h(r) = v. \quad (4)$$

By requiring that the solution is non-singular at the origin, we also have

$$\left. \frac{dh(r)}{dr} \right|_{r=0} = 0. \quad (5)$$

Eq. (1) gives the leading contribution to the tunnelling rate in the semiclassical limit, that is, it only includes exponentially enhanced terms in the limit $\hbar \rightarrow 0$. In particular, the overall normalization can only be determined by including the first quantum corrections [12].

We observe that the bounce is the unique solution of a suitably defined Cauchy problem. Indeed, under the assumption that the bounce is $O(4)$ -invariant, the field equation for the bounce is

$$h''(r) = -\frac{3}{r} h'(r) + V'(h), \quad (6)$$

where $V(h)$ is the scalar potential, and primes denote differentiation with respect to the functional argument. A set of initial conditions

$$h(r_0) = h_0; \quad h'(r_0) = h_1, \quad (7)$$

given at any finite value $r_0 \neq 0$, defines a Cauchy problem, which has a unique solution in a neighbourhood of r_0 , as a consequence of known results in real analysis. Since $r = 0$ is the only singular point in the r.h.s. of Eq. (6), r_0 can be taken to be arbitrarily large. However, the unique solution is not necessarily well defined at $r = 0$, because of the singularity in the r.h.s. of Eq. (6), nor at $r \rightarrow \infty$, because the existence and unicity theorem has a local meaning. Thus, for a generic choice of the initial conditions at $r = r_0$, the boundary conditions Eqs. (4), (5), are not necessarily fulfilled. Conversely, the requirement that Eqs. (4), (5) are fulfilled by the bounce selects a set of allowed initial conditions

at an intermediate point r_0 . Depending on the shape of the potential, it is possible to have multiple bounce solutions (with different initial values h_0 and h_1) which satisfy the boundary conditions in Eqs. (4), (5).

2.1. Decay of the SM vacuum in the semiclassical approximation

The case of the pure SM is especially interesting. In this case, the value of the true vacuum is typically very large with respect to the electroweak scale $v \approx 246$ GeV, so one usually takes $v = 0$. Thus

$$\lim_{r \rightarrow \infty} h(r) = 0. \quad (8)$$

The validity of this approximation is discussed below in Sect. 2.5. The scalar potential in the unstable region is therefore

$$V(h) = \frac{1}{4} \lambda h^4, \quad (9)$$

where we neglect the logarithmic running of λ and take it as a negative constant. This approximation is reviewed in Sect. 2.2. Then Eq. (6) takes the form

$$h''(r) + \frac{3}{r} h'(r) = \lambda h^3(r). \quad (10)$$

Eq. (10) is invariant under scale transformations: if $h(r)$ is a solution, then

$$h_a(r) = ah(ar) \quad (11)$$

is also a solution, for any choice of the scale factor a . Indeed

$$\begin{aligned} h_a''(r) + \frac{3}{r} h_a'(r) &= a^3 \left[h''(ar) + \frac{3}{ar} h'(ar) \right] \\ &= \lambda a^3 h^3(ar) = \lambda h_a^3(r). \end{aligned} \quad (12)$$

Obviously, the scaled solution $h_a(r)$ has the same limiting behaviors Eqs. (4), (5) as the original one, but different initial conditions at $r = r_0$. Otherwise stated, the boundary conditions do not fix the overall normalization of the bounce solution.

As is well known, a solution of Eq. (10) with the boundary conditions (4), (5) is given by the Fubini–Lipatov instanton [14,15]

$$h(r) = \sqrt{\frac{8}{|\lambda|}} \frac{R}{R^2 + r^2}, \quad (13)$$

for any value of R and $\lambda < 0$ (note that $h(R) = h(0)/2$: this will be our definition of the size of the bounce throughout the paper). It should be clear from the above discussion that the presence of the arbitrary parameter R is just a reflection of the scale invariance of the equation, and not a signal of non-unicity of the solution; indeed, the scaling defined in Eq. (11) amounts to replacing R with R/a in Eq. (13). $h(r)$ is the unique solution of Eq. (6) with initial conditions in r_0

$$h_0 = \sqrt{\frac{8}{|\lambda|}} \frac{R}{R^2 + r_0^2}; \quad h_1 = -\sqrt{\frac{8}{|\lambda|}} \frac{2Rr_0}{(R^2 + r_0^2)^2}, \quad (14)$$

which obey the constraints (4), (5).

The SM bounce Eq. (13) can be found by the following procedure (see e.g. Ref. [16] for an alternative derivation). Let us assume that a solution of Eq. (10) exists, with a Taylor expansion around $r = 0$:

$$h(r) = \sum_{k=0}^{\infty} A_k r^k \quad (15)$$

with $A_0 > 0$. Eq. (10) takes the form

$$\frac{3A_1}{r} + \sum_{k=0}^{\infty} (k+2)(k+4)A_{k+2}r^k = \lambda \sum_{k=0}^{\infty} r^k \sum_{i=0}^k \sum_{j=0}^{k-i} A_i A_j A_{k-i-j}. \quad (16)$$

It follows that

$$A_1 = 0. \quad (17)$$

Thus, the condition that the first derivative of the bounce at $r = 0$ vanishes is a consequence of the assumption Eq. (15). The remaining coefficients are given by the recurrence relation

$$A_{k+2} = \frac{\lambda}{(k+2)(k+4)} \sum_{i=0}^k \sum_{j=0}^{k-i} A_i A_j A_{k-i-j}. \quad (18)$$

The coefficients A_k with k odd are zero: indeed, when k is odd, one out of the three summation indices $i, j, k-i-j$ is also odd (possibly all of them). Hence $A_1 = 0$ implies $A_3 = 0$, and so on. Thus, we may rewrite Eq. (15) as

$$h(r) = \sum_{k=0}^{\infty} a_k r^{2k}; \quad a_k = A_{2k}, \quad (19)$$

and the recurrence relation for the coefficients becomes

$$a_{k+1} = \frac{\lambda}{8} \frac{2}{(k+1)(k+2)} \sum_{i=0}^k \sum_{j=0}^{k-i} a_i a_j a_{k-i-j}. \quad (20)$$

The coefficients a_k are determined by the single number a_0 , the value of the bounce at the origin. We now show that

$$a_j = \left(\frac{\lambda}{8}\right)^j a_0^{2j+1}. \quad (21)$$

The proof is by induction. For $k = 0$ Eq. (20) gives

$$a_1 = \frac{\lambda}{8} a_0^3. \quad (22)$$

We now assume that Eq. (21) holds for $0 \leq j \leq k$. Then

$$\begin{aligned} a_{k+1} &= \frac{\lambda}{8} \frac{2}{(k+1)(k+2)} \sum_{i=0}^k \sum_{j=0}^{k-i} a_i a_j a_{k-i-j} \\ &= \frac{\lambda}{8} \frac{2}{(k+1)(k+2)} \sum_{i=0}^k \sum_{j=0}^{k-i} \left(\frac{\lambda}{8}\right)^k a_0^{2k+3} \\ &= \left(\frac{\lambda}{8}\right)^{k+1} a_0^{2k+3} \end{aligned} \quad (23)$$

which is what we set out to prove. The Taylor expansion in Eq. (19) can now be summed. We find

$$h(r) = a_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{8}\right)^k a_0^{2k} r^{2k} = \frac{a_0}{1 - \frac{\lambda}{8} a_0^2 r^2}. \quad (24)$$

The series has convergence radius

$$R = \sqrt{\frac{8}{|\lambda|} \frac{1}{a_0}}, \quad (25)$$

but if λ is negative the sum can be analytically continued to the whole positive real axis, and vanishes as $r \rightarrow \infty$. Finally, we note that for $\lambda < 0$ the bounce in Eq. (24) coincides with the solution

given in Eq. (13), with R as in Eq. (25). The above construction shows that, given the value of the bounce at the origin and the requirement of regularity on the range $0 < r < \infty$, the solution is unique.

The value of the euclidean action $S[h]$ of the bounce solutions Eq. (13) is

$$S[h] = \frac{8\pi^2}{3|\lambda|}, \quad (26)$$

independently of the value of R . This is not surprising, because the action is dimensionless in natural units, and no dimensionfull scale parameter is available other than R . Hence, there is no way to single out one preferred value of R at the semiclassical level. However, R is related to the value of the bounce at $r = 0$:

$$h(0) = \sqrt{\frac{8}{|\lambda|} \frac{1}{R}}, \quad (27)$$

and since the bounce solution only exists for $\lambda < 0$, we expect that

$$\frac{1}{R} > \Lambda_I, \quad (28)$$

where $\Lambda_I \approx 10^{10}$ GeV is the energy scale at which the running coupling $\lambda(\mu)$ becomes negative. This is an *a posteriori* confirmation that neglecting the electroweak scale, of order 10^2 GeV, with respect to the size of the bounce is indeed a reliable approximation.

2.2. Violation of scale invariance through radiative corrections

The first quantum corrections, computed in Ref. [12], affect the semiclassical result in two respects: they fix the normalization in Eq. (1), and they take into account the running of the Higgs coupling λ .¹ As a consequence, the tunnelling decay rate is dominated by the bounce with the maximum value of $|\lambda(\mu)| \sim |\lambda(1/R_{\text{SM}})|$. For the central values of the SM parameters the scale $1/R_{\text{SM}}$ turns out to be a couple of order of magnitudes below the Planck mass $M_P = 1.22 \times 10^{19}$ GeV.

Schematically, one expands the euclidean action around the tree-level bounce solution

$$S[h + \tilde{h}] \approx S[h] + \frac{1}{2} \int d^4x S''[h] \tilde{h}^2, \quad (29)$$

and integrates over the fluctuations \tilde{h} :

$$\int \mathcal{D}[\tilde{h}] e^{-S[h+\tilde{h}]} \approx e^{-S[h]} (\det S''[h])^{-\frac{1}{2}}. \quad (30)$$

In the treatment of the functional determinant there are two main aspects which eventually lead to the appearance of the scale $\mu \sim 1/R_{\text{SM}}$ in the calculation: *i*) ultraviolet (UV) divergences of the non-zero modes of $S''[h]$, responsible for the introduction of a renormalization scale μ and *ii*) the treatment of the zero modes of $S''[h]$, which have to be singled out and treated separately in order to avoid unphysical divergences. The existence of zero modes is simply a reflection of the fact that the classical action is invariant under a larger class of symmetries which are broken by the explicit solution of the tree-level bounce (e.g. in the case of Eq. (13) these are translations in $O(4)$ and scale transformations). The symmetries of the action are hence restored only if one considers the

¹ For some earlier works discussing the breaking of scale invariance due to the running of λ in the context of the SM-vacuum tunnelling calculation see also Refs. [17,18].

family of all bounce solutions as a whole. This defines a measure of integration (the instanton measure $d\mu_{\text{inst}}$ [19]) over the so-called collective coordinates, parametrizing the families of bounces with different sizes (R) and located anywhere in euclidean spacetime (x_0)

$$d\mu_{\text{inst}} \approx \frac{dR d^4 x_0}{R^5} \exp \left[-\frac{8\pi^2}{3|\lambda(1/R)|} - \Delta S \right], \quad (31)$$

where $\lambda(1/R)$ is the renormalized coupling at the scale $1/R$, $\Delta S = \mathcal{O}(g_i^2/\lambda)$ (with i running over the SM couplings) comes from the inclusion of the non-zero modes and contains finite terms plus logs which are minimized by the choice $\mu \sim 1/R$ [12]. The tunnelling probability is therefore obtained by integrating over the collective coordinates

$$p = \int d\mu_{\text{inst}} \approx \tau_U^4 \int \frac{dR}{R^5} \exp \left[-\frac{8\pi^2}{3|\lambda(1/R)|} - \Delta S \right], \quad (32)$$

where the integral is extended to the range where $\lambda(1/R) < 0$, where a bounce exists.

By expanding $\lambda(1/R) < 0$ around its minimal value located at $R = R_{\text{SM}}$

$$\lambda(1/R) \approx \lambda(1/R_{\text{SM}}) + \frac{1}{2} \lambda''(1/R_{\text{SM}}) (1/R - 1/R_{\text{SM}})^2, \quad (33)$$

we can approximate the integral by the method of the steepest descent

$$p \approx \frac{\tau_U^4}{R_{\text{SM}}^3} \sqrt{\frac{3\lambda^2(1/R_{\text{SM}})}{4\pi\lambda''(1/R_{\text{SM}})}} \exp \left[-\frac{8\pi^2}{3|\lambda(1/R_{\text{SM}})|} - \Delta S \right] \approx \frac{\tau_U^4}{R_{\text{SM}}^4} \exp \left[-\frac{8\pi^2}{3|\lambda(1/R_{\text{SM}})|} \right], \quad (34)$$

where in the last step we have neglected the subleading pre-exponential factors (including ΔS), and we have set $\lambda''(1/R_{\text{SM}}) \sim R_{\text{SM}}^2$ on dimensional grounds.

The result Eq. (34) provides the standard approximation which is usually employed in order to estimate the lifetime of the electroweak vacuum. Nonetheless, it is interesting to compare it with a more direct calculation obtained via the renormalization group (RG) improved effective potential

$$V_{\text{eff}} = \frac{1}{4} \lambda_{\text{eff}}(h) h^4 \approx \frac{1}{4} \lambda(h) h^4, \quad (35)$$

where, as in Eq. (9), we neglected the mass term and in the last step we have approximated the effective quartic coupling λ_{eff} with the $\overline{\text{MS}}$ renormalized coupling λ evaluated at $\mu = h$. The scale ambiguity of the tree-level bounce is now resolved by the leading order effective potential which takes into account the dominant radiative corrections. However, since it is not possible to find an analytical solution for the bounce one has to resort to a numerical analysis (see Appendix A for details). By applying our numerical set-up² we obtain the bounces displayed in Fig. 1 where, for illustrative purposes, we considered the case where the running of λ is evaluated at one (1ℓ), two (2ℓ) and three (3ℓ) loops, respectively. The corresponding bounce actions are found to be

$$S[h, \lambda(h)^{1\ell}] = 772.3, \quad (36)$$

$$S[h, \lambda(h)^{2\ell}] = 1703.9, \quad (37)$$

$$S[h, \lambda(h)^{3\ell}] = 1788.8. \quad (38)$$

² In this work we refer to arXiv v4 of [2] for the central values of the SM couplings evaluated at the top-quark pole-mass scale. In particular, we take $m_t = 173.34$ GeV and $m_h = 125.15$ GeV.

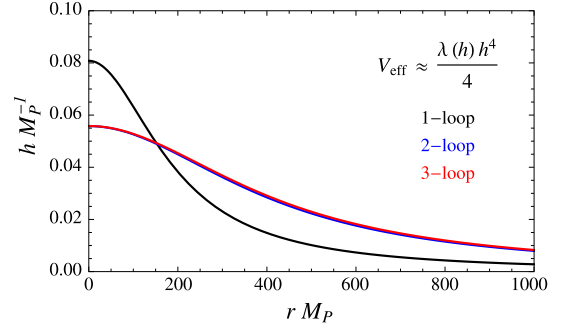


Fig. 1. Bounce profiles for the potential in Eq. (35), where the running of λ is evaluated at one (black), two (blue) and three (red) loops, respectively. The adimensional variables are normalized in terms of the Planck mass M_P . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

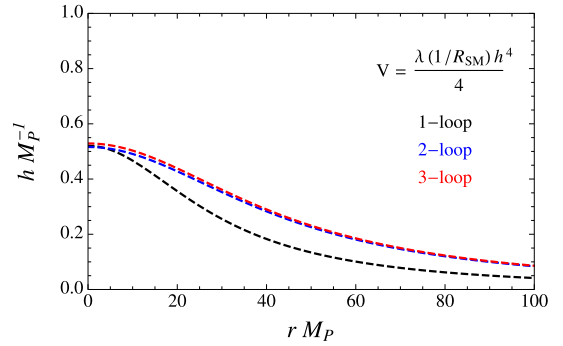


Fig. 2. Same as in Fig. 1 for the SM tree-level potential with $\lambda(1/R_{\text{SM}})$ a negative constant.

On the other hand, we want to compare the latter case with the standard method (cf. Eq. (34)) where λ is evaluated at the scale $1/R_{\text{SM}}$ where $\beta_\lambda \equiv \frac{d\lambda}{d \log \mu} = 0$. For our set of parameters, we get³

$$1/R_{\text{SM}}^{1\ell} = 4.145 \times 10^{17} \text{ GeV}; \quad \lambda(1/R_{\text{SM}})^{1\ell} = -0.03409,$$

$$1/R_{\text{SM}}^{2\ell} = 2.768 \times 10^{17} \text{ GeV}; \quad \lambda(1/R_{\text{SM}})^{2\ell} = -0.01546,$$

$$1/R_{\text{SM}}^{3\ell} = 2.769 \times 10^{17} \text{ GeV}; \quad \lambda(1/R_{\text{SM}})^{3\ell} = -0.01473,$$

which yields the bounces displayed in Fig. 2. Correspondingly, the bounce actions defined by

$$S[h, \lambda(1/R_{\text{SM}})] = \frac{8\pi^2}{3|\lambda(1/R_{\text{SM}})|} \quad (39)$$

have the following values

$$S[h, \lambda(1/R_{\text{SM}})^{1\ell}] = 772.0, \quad (40)$$

$$S[h, \lambda(1/R_{\text{SM}})^{2\ell}] = 1702.6, \quad (41)$$

$$S[h, \lambda(1/R_{\text{SM}})^{3\ell}] = 1787.4. \quad (42)$$

Although the profiles of the bounces in Fig. 1 and Fig. 2 look different, the bounce actions in Eqs. (36)–(38) and Eqs. (40)–(42) are very similar.⁴ In particular, Eq. (39) provides a lower bound on the bounce action, since it is obtained by expanding $\lambda(1/R)$ around its minimum.

³ By taking into account the 3σ error bands of the most important SM parameters there is about one order of magnitude uncertainty on the scale where $\beta_\lambda = 0$ [2].

⁴ This fact can be understood as follows: the bounces in Fig. 2 can be rescaled by $h(r) \rightarrow ah(ar)$ in such a way that they almost superimpose with those in Fig. 1. The corresponding change in the bounce action, induced by the rescaling $R_{\text{SM}} \rightarrow R_{\text{SM}}/a$, is small, because λ varies very slowly around its minimum.

To conclude, the good agreement between the two procedures for the determination of the decay probability justifies the approximation made in [12] of taking a constant $\lambda < 0$ for the leading order SM bounce. This was not completely obvious a priori, since computing the bounce “does not commute” with the running of λ . Of course, the potential with running λ in Eq. (35) only captures log-enhanced corrections to the tunnelling rate, while a complete one-loop calculation requires the determination of the SM action functional around the leading order bounce configuration [12].

2.3. Gauge independence of the tunnelling rate

A question which is directly related to the calculation of quantum corrections to the SM vacuum decay rate is that of gauge invariance. Indeed, if one naively takes into account loop corrections to the tunnelling rate by computing the bounce via the RG improved effective potential in Eq. (35) the result will look gauge dependent. The dependence on the gauge-fixing parameters in λ_{eff} (e.g. in the Fermi gauge [20]) is two-fold: it originates from the fixed-order expression of the effective potential, and from its running via the anomalous dimension of the field h .

From this point of view, gauge dependence (even if numerically small) is a good thing, since is telling us that we are computing something in the wrong way. As shown in Ref. [12], the divergent corrections to the bounce action are formally gauge independent, being directly related to the beta function of λ . A crucial role in order to achieve the cancellation of the gauge dependent parts is played by the kinetic part of the effective action, which is neglected altogether when dealing with the effective potential only.

More generally, the gauge independence of the tunnelling rate directly follows from the Nielsen identity [21,22]

$$\xi \frac{\partial S_{\text{eff}}}{\partial \xi} = \int d^4x \frac{\delta S_{\text{eff}}}{\delta h(x)} K[h(x)], \quad (43)$$

where ξ denotes the gauge-fixing parameter and K is a functional of h whose expression depends on the gauge fixing.

The physical implication of Eq. (43) is clear: the effective action is gauge independent when evaluated on a configuration which extremizes it. Hence, the bounce action is formally gauge independent as well.

The proof of the gauge independence of the tunnelling rate can be carried out in perturbation theory by means of a loop expansion of the Nielsen identity, along the lines of Ref. [23]. In practice, however, the cancellation of the gauge dependent parts in an explicit calculation might require some care. As recently observed in Refs. [24,25], the usual loop expansion is not the consistent one for the SM, where $\lambda \sim \hbar$ (as in the original Coleman–Weinberg (CW) model [26]) in order for the top–Yukawa corrections to destabilize the tree-level electroweak vacuum. Consequently, such a modified loop expansion must be properly taken into account in order to observe the gauge independence of the SM tunnelling rate in perturbation theory. In particular, this entails the resummation of a particular class of daisy diagrams which (as observed in Ref. [27]) is connected with the resummation of IR-divergent Goldstone loops [28,29].

Finally, we notice that for CW-like potentials, where the absolute minimum is radiatively generated (as in the SM), the standard bounce formalism requires some modifications [30]. On the other hand, given the fact that for the measured values of the SM parameters the lifetime of the electroweak vacuum turns out to be much larger than the age of the Universe, precision calculations of the SM tunnelling rate, although important, are not crucial at the moment.

2.4. Tunnelling without barriers

The approximation of taking λ a negative constant might still appear rather odd, since it corresponds to a tunnelling process from a potential of the type λh^4 , with no barriers and a maximum in $h = 0$. It is well known, however, that the absence of a barrier in the scalar potential is not necessarily a problem in field theory, due to the presence of an extra barrier originated by the gradient of the bounce [16]. In this Section we want to explicitly check this statement in the case of the SM.

The tunnelling process in field theory has to be understood as a transition between two spatial field configurations at different euclidean times τ_i and τ_f

$$\lim_{\tau_i \rightarrow -\infty} h(\vec{x}, \tau_i) = v; \quad \lim_{\tau_f \rightarrow +\infty} h(\vec{x}, \tau_f) = v, \quad (44)$$

where v denotes the false vacuum. The bounce action entering the expression of the tunnelling probability in Eq. (1) can be recast in a way that resembles the analogous one in quantum mechanics [16]

$$S[h] = \int_{\tau_i}^{\tau_f} d\tau K(\tau) \sqrt{2U[h]}, \quad (45)$$

where the factor

$$K(\tau) = \left[\int d^3x \left(\frac{\partial h}{\partial \tau} \right)^2 \right]^{\frac{1}{2}} \quad (46)$$

yields the correct normalization of the path length, and

$$U[h] = \int d^3x \left[\frac{1}{2} \left(\vec{\nabla} h(\vec{x}, \tau) \right)^2 + V(h(\vec{x}, \tau)) \right] \quad (47)$$

plays the role of the potential energy as in ordinary quantum mechanics.

It is an instructive exercise to verify the existence of an actual barrier in the case of the SM potential with $\lambda < 0$. For simplicity (and in order to proceed analytically) we take the mass parameter $m = 0$ in the scalar potential. Starting from the $O(4)$ -invariant bounce solution Eq. (13), a straightforward calculation yields

$$T[h] \equiv \int d^3x \frac{1}{2} \left(\vec{\nabla} h(\vec{x}, \tau) \right)^2 = \frac{2\pi^2}{|\lambda| R} \left(\frac{1}{1 + \frac{\tau^2}{R^2}} \right)^{\frac{3}{2}}, \quad (48)$$

$$V[h] \equiv \int d^3x V(h(\vec{x}, \tau)) = -\frac{2\pi^2}{|\lambda| R} \left(\frac{1}{1 + \frac{\tau^2}{R^2}} \right)^{\frac{5}{2}}, \quad (49)$$

and

$$K(\tau) = \frac{2\pi}{\sqrt{|\lambda| R}} \frac{\frac{\tau}{R}}{\left(1 + \frac{\tau^2}{R^2} \right)^{\frac{5}{4}}}. \quad (50)$$

Following then the definitions in Eq. (47) and Eq. (45), we finally get

$$U[h] = \frac{2\pi^2}{|\lambda| R} \left(\frac{1}{1 + \frac{\tau^2}{R^2}} \right)^{\frac{3}{2}} \left[1 - \frac{1}{1 + \frac{\tau^2}{R^2}} \right], \quad (51)$$

and

$$S[h] = 2 \int_0^\infty d\tau K(\tau) \sqrt{2U[h]} = \frac{8\pi^2}{3|\lambda|}, \quad (52)$$

which reproduces the correct result for the SM bounce action.

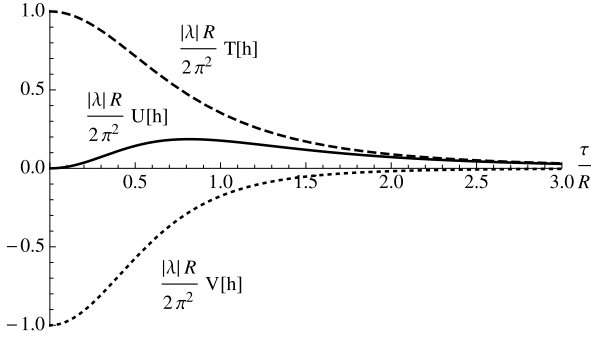


Fig. 3. Shapes of the gradient ($T[h]$) and potential ($V[h]$) barriers and their sum ($U[h]$) as a function of $\frac{\tau}{R}$.

The three quantities $T[h]$, $V[h]$ and $U[h]$ are plotted in Fig. 3. $U[h]$ as a function of τ can be interpreted as the potential energy along the path which minimizes the euclidean action. Thanks to the positive gradient contribution $T[h]$ we see that there is a barrier even for λ constant and negative. Notice the correct asymptotic behavior of $U[h]$, which tends to zero as $\tau \rightarrow \infty$ since h is approaching the false vacuum $v \rightarrow 0$. On the other hand, $U[h] = 0$ for $\tau = 0$ corresponds to complete barrier penetration. Since the point $\tau = 0$ coincides with $t = 0$ ($t = -i\tau$) the bounce solution can be analytically continued in Minkowski space, so that the system evolves towards the true minimum following the classical equation of motion.

From the above discussion one can also draw another important conclusion: the largest energy scale relevant for barrier penetration is not that for which $V = 0$ (i.e. the instability scale of the SM effective potential), but rather the value of the bounce in its center $h(\tau = 0)$ which corresponds to zero potential energy U .

2.5. Violation of scale invariance by mass terms

In the previous sections (as in most of the existing literature on this subject) the lifetime of the metastable vacuum of the SM was computed neglecting the mass term of the Higgs boson, on the basis that the electroweak scale, of order 10^2 GeV, is much smaller than $1/R$, where R is the typical size of the relevant bounce. We now wish to discuss this approximation in some detail.

Let us consider the action

$$S[h] = \int d^4x \left[\frac{1}{2} \partial_\mu h \partial_\mu h + \frac{1}{2} m^2 h^2 + \frac{1}{4} \lambda h^4 \right], \quad (53)$$

where $m^2 > 0$ and $\lambda < 0$. It was pointed out long ago [31] that a bounce in this case does not exist. The easiest way to see this is to perform a scale transformation, defined in Eq. (11). We get

$$S[h] \rightarrow S[h_a] = S[h] + \frac{m^2}{2a^2} \int d^4x h^2(x), \quad (54)$$

which cannot be stationary upon scale transformations unless $h = 0$:

$$\left. \frac{\partial S[h_a]}{\partial a} \right|_{a=1} = -m^2 \int d^4x h^2(x). \quad (55)$$

This phenomenon is well known in the context of studies of instanton gauge field configurations.

Nevertheless, it is reasonable to think that, even in the presence of a mass term, instanton configurations of the scalar fields should exist, provided they are characterized by a length scale R such that $m \ll 1/R$. Roughly speaking, such a solution of the field equation is expected to be a function of r , approximately constant for $r < R$

and approximately zero for $r > R$. Furthermore, if no other mass scale is available, the value of the bounce for $r < R$, $h(0)$, is proportional to $1/R$ for dimensional reasons. Hence, the characteristic scale of h may be identified by the spacetime integral of a local operator, function of h . For example,

$$\int d^4x h^n(x) \sim h^n(0) \int_0^R r^3 dr \sim R^{4-n}. \quad (56)$$

Based on these intuitive considerations, it is suggested in Ref. [31] to perform a minimization of the action functional in which the minimum configuration is constrained to be characterized by a scale R much smaller than $1/m$. The constraint is introduced by means of a suitable Lagrange multiplier σ , i.e. by adding to the action a term

$$S_c[h] = \sigma \left[\int d^4x O(h) - c R^{4-n} \right], \quad (57)$$

where $O(h)$ is a local operator of mass dimension $n \neq 4$, for example $O(h) = h^6$, and c a constant. It is shown that with this modification an instanton appears, called a *constrained instanton*, which differs from the bounce of the massless theory by powers of $m^2 R^2$, times possibly logs of mR . Thus, the corrections to the usually adopted approximation are indeed small, provided $mR \ll 1$.

The mechanism which restores the existence of a bounce in the massive theory is illustrated in the Appendix of Ref. [31] for the choice $O(h) = h^6$. The key point is that the constraint Eq. (57), with $\sigma > 0$, has the effect of generating an absolute minimum (the true vacuum) of the scalar potential, which would be unbounded from below with $\sigma = 0$ and $\lambda < 0$. Explicitly, the new scalar potential

$$V_c(h) = \frac{1}{2} m^2 h^2 + \frac{\lambda}{4} h^4 + \sigma h^6 \quad (58)$$

with $\lambda < 0$, $m^2 > 0$ has a local minimum at $h = 0$, with $V(0) = 0$, and an absolute minimum at $h \approx \sqrt{\frac{|\lambda|}{6\sigma}}$ (for $m^2 \sigma \ll 1$). The presence of the constraining term locally restores the scale invariance of the action:

$$\begin{aligned} \left. \frac{\partial}{\partial a} (S[h_a] + S_c[h_a]) \right|_{a=1} &= -m^2 \int d^4x h^2(x) + 2\sigma \int d^4x h^6(x) \\ &= -m^2 \int d^4x h^2(x) + \frac{2c\sigma}{R^2}, \end{aligned} \quad (59)$$

which is zero for

$$\sigma = \frac{m^2 R^2}{2c} \int d^4x h^2(x) \sim (mR)^2 R^2 \ll R^2. \quad (60)$$

This issue is not directly relevant in the SM; indeed, as we have seen in Sect. 2.2, scaling violation induced by radiative corrections have the effect of selecting a bounce of size $R_{\text{SM}} \sim 10^{-17}$ GeV $^{-1}$. Explicitly, with

$$V(h) = \frac{1}{4} \lambda(h) h^4 \quad (61)$$

we have

$$\left. \frac{\partial}{\partial a} S[h_a] \right|_{a=1} = \frac{1}{4} \int d^4x \beta_\lambda(h) h^4(x), \quad (62)$$

and $\beta_\lambda(\mu)$ is zero around $\mu = 1/R_{\text{SM}} \sim 10^{17}$ GeV. As a consequence, scale invariance is locally restored, and a bounce of size $R \sim R_{\text{SM}}$ is found. The effects of quadratic and cubic terms in the potential are suppressed by powers of $v R_{\text{SM}} \sim 10^{-15}$ and can be

safely neglected. Nevertheless, the simple example discussed above shows that the explicit violation of scale invariance (for example, by the introduction of non-renormalizable terms in the scalar potential) may lead to complications in the calculation of the vacuum tunnelling rate.

3. Violation of scale invariance by new interactions

We now turn to a discussion of the effects of SM extensions characterized by the appearance of an explicit new energy scale in the effective potential. Following Refs. [6–8], we describe the effect of generic NP occurring at high-energy scales by introducing two non-renormalizable terms in the scalar potential,

$$V(h) = \frac{1}{4}\lambda h^4 + \frac{\lambda_6}{6M^2}h^6 + \frac{\lambda_8}{8M^4}h^8. \quad (63)$$

Assuming the new couplings ($\lambda_{6,8}$) to be of order one, the extra terms affect the potential only for field values of order M . The scalar potential in Eq. (63) should be understood just as a toy model with no claim of being realistic. It is however sufficient in order to highlight some basic features of the tunnelling rate in the presence of new interactions. A more realistic expression of the modified potential in presence of NP is obtained by considering a full tower of higher-dimensional operators, namely

$$V(h) = \frac{1}{2}\lambda_2 M^2 h^2 + \frac{1}{4}\lambda h^4 + \sum_{n=3}^{\infty} \frac{\lambda_{2n}}{2nM^{2n-4}} h^{2n}, \quad (64)$$

where we have also inserted a mass term due to the presence of a physical threshold at the scale M ,⁵ assuming that, by a suitable fine-tuning, the electroweak scale is kept at the correct phenomenological value.

Most of the following discussion and results apply to the general potential in Eq. (64), the key point being the explicit breaking of scale invariance characterized by the scale M .⁶ In principle, effective operators involving derivatives of the Higgs field should also be included, since they influence the determination of the bounce as well. By employing a suitable field redefinition, it can be shown [32] that they can affect the bounce only when their dimension d exceeds $d = 6$.

In Refs. [6–8] the mass scale M , required by the presence of non-renormalizable terms, is taken to be of the order of the Planck mass. Here we postpone for a while the choice of a definite value for M , keeping the discussion on general terms. We will however be especially interested in the case $M \gg 1/R_{\text{SM}}$, since for $M \lesssim 1/R_{\text{SM}}$ the SM bounce gets trivially modified and the lifetime of the electroweak vacuum can be any with respect to the SM case.

Because of the dependence on M of the effective potential, the action is no longer scale invariant. Hence, the existence of a bounce for generic choices of the new couplings λ_{2n} is no longer guaranteed. For example, with all $\lambda_{2n} = 0$ except λ_6 the action can only be stationary upon scale transformations if $h = 0$. We have indeed

$$S[h] \rightarrow S[h_a] = S[h] + \frac{\lambda_6 a^2}{6M^2} \int d^4x h^6(x), \quad (65)$$

which implies

$$\left. \frac{\partial S[h_a]}{\partial a} \right|_{a=1} = \frac{\lambda_6}{3M^2} \int d^4x h^6(x). \quad (66)$$

⁵ We thank Ulrich Ellwanger for pointing out this effect to us.

⁶ The whole discussion applies as well to the case where such effective interactions arise within an explicit renormalizable model with new degrees of freedom, such as the one proposed in Ref. [9].

However, in analogy with the case of a mass term discussed in Section 2.5, one may argue that a constrained instanton with $R \gg 1/M$ may still exist. Furthermore, if more non-renormalizable couplings are different from zero (e.g. both $\lambda_6 \neq 0$ and $\lambda_8 \neq 0$), the r.h.s. of Eq. (66) contains more terms. In such case one can conceive the possibility that these terms compensate each other (assuming some of the couplings have opposite signs) for instanton solutions characterized by $R \sim 1/M$. On general grounds, we thus expect two classes of instanton solutions (when they exist): those characterized by $R \gg 1/M$ and those characterized by $R \sim 1/M$.

With a generic potential of the type of Eq. (64), the field equation cannot be integrated analytically, and one has to adopt numerical methods (which we are going to do below). However, a somewhat deeper understanding of the tunnelling process in the presence of NP can be achieved by making contact with the case of the pure SM. To this purpose, it is useful to rewrite the modified potential Eq. (64) in terms of an effective quartic coupling⁷:

$$V(h) = \frac{1}{4}\lambda_{\text{eff}}(h)h^4, \quad (67)$$

with

$$\lambda_{\text{eff}}(h) = \frac{2\lambda_2 M^2}{h^2} + \lambda + 4 \sum_{n=3}^{\infty} \frac{\lambda_{2n}}{2nM^{2n-4}} h^{2n-4}, \quad (68)$$

where, unless otherwise specified, we take the quartic coupling λ to be at its minimal negative value (as in the SM case) and we neglect the logarithmic running for the NP couplings, since their are anyway unknown. Note that the coupling λ_2 becomes active only above threshold $h \gtrsim M$.

If we were able to argue that the dominant contribution to the bounce action is provided by taking the argument of λ_{eff} at some fixed value, then the action would be approximately given by the SM expression Eq. (26), with λ replaced by the appropriate value of λ_{eff} .

It can be seen that this is indeed the case by adopting the approximation mentioned in Section 2.5, namely, to consider the bounce as a finite-action, $O(4)$ -invariant solution of the euclidean field equation $h(r)$, approximately constant for $r < R$ and approximately zero for $r > R$. Let us first test this approximation in the case of the SM with no mass term, where the exact result is known. In this case, the constant value $h(0)$ of the bounce for $0 < r < R$ must be proportional to $1/(R\sqrt{|\lambda|})$. The factor $1/R$ arises for dimensional reasons. The factor $1/\sqrt{|\lambda|}$ can be understood by writing the action as

$$\begin{aligned} S_{\text{SM}}[h] &= \int d^4x \left[\frac{1}{2} \partial_\mu h \partial_\mu h + \frac{1}{4} \lambda h^4 \right] \\ &= \frac{1}{\lambda} \int d^4x \left[\frac{1}{2} \partial_\mu H \partial_\mu H + \frac{1}{4} H^4 \right] \end{aligned} \quad (69)$$

where the bounce

$$H(x) = \sqrt{\lambda} h(x) \quad (70)$$

does not depend on λ .

Within this approximation we get

$$\begin{aligned} S_{\text{SM}}[h] &= 2\pi^2 \int_0^\infty dr r^3 \left[V(h) - \frac{1}{2} h(r) V'(h) \right] \\ &\approx \frac{\pi^2 R^4}{2} \left[V(h(0)) - \frac{1}{2} h(0) V'(h(0)) \right] \end{aligned}$$

⁷ Note that this parametrization is not directly related to the effective potential.

$$\begin{aligned}
&= \frac{\pi^2 R^4}{8} |\lambda| h^4(0) \\
&= \frac{a^4 \pi^2}{8 |\lambda|}, \quad (71)
\end{aligned}$$

where a is a constant, and we have used the field equation in the first equality. The correct value $S_{\text{SM}}[h] = 8\pi^2/(3|\lambda|)$ is reproduced for $a^2 = 8/\sqrt{3} \approx (2.15)^2$.

We now turn to the case of the modified potential Eq. (64). We observe that, within the same approximation that led us to Eq. (71), the effect of a scale transformation Eq. (11) on the modified action is

$$\left. \frac{\partial S[h_a]}{\partial a} \right|_{a=1} = \int d^4x \frac{1}{4} \frac{\partial \lambda_{\text{eff}}(h)}{\partial h} h^5(x) \sim h(0) \lambda'_{\text{eff}}(h(0)). \quad (72)$$

Hence, in order for the action to be stationary under scale transformations, $h(0)$ should be chosen so that

$$\lambda'_{\text{eff}}(h(0)) = 0 \quad (73)$$

locally restores scale invariance in a neighbourhood of $h(0)$ (the case $h(0) = 0$ is obviously uninteresting). If such a value of $h(0)$ exists, then

$$\begin{aligned}
S[h] &= 2\pi^2 \int_0^\infty dr r^3 \left[V(h) - \frac{1}{2} h(r) V'(h) \right] \\
&\approx \frac{\pi^2 R^4}{2} \left[V(h(0)) - \frac{1}{2} h(0) V'(h(0)) \right] \\
&= \frac{\pi^2 R^4}{2} \left[-\frac{1}{4} \lambda_{\text{eff}}(h(0)) h^4(0) - \frac{1}{8} h^5(0) \lambda'_{\text{eff}}(h(0)) \right] \\
&= -\frac{\pi^2 R^4}{8} \lambda_{\text{eff}}(h(0)) h^4(0), \quad (74)
\end{aligned}$$

which is precisely what we would get in the SM with no mass term, and λ replaced by $\lambda_{\text{eff}}(h(0))$, see the third line in Eq. (71). We conclude that the leading contribution to the bounce action is given by

$$S[h] \approx \frac{8\pi^2}{3|\lambda_{\text{eff}}(h(0))|}, \quad (75)$$

provided

$$h(0) = \frac{a}{R\sqrt{|\lambda_{\text{eff}}(h(0))|}}, \quad a = \sqrt{\frac{8}{\sqrt{3}}}. \quad (76)$$

As an example, let us consider the potential in Eq. (63). We have in this case

$$\lambda_{\text{eff}}(h) = \lambda + \frac{2}{3} \lambda_6 \frac{h^2}{M^2} + \frac{1}{2} \lambda_8 \frac{h^4}{M^4}, \quad (77)$$

$$\lambda'_{\text{eff}}(h) = \frac{2}{3} \frac{h}{M^2} \left(2\lambda_6 + 3\lambda_8 \frac{h^2}{M^2} \right). \quad (78)$$

We can distinguish three basic scenarios.

- I. One possibility is that both λ_6 and λ_8 are non-negative and at least one of them is different from zero. In such case the potential is bounded from below and, for any value of h , $\lambda_{\text{eff}}(h) > \lambda(h)$.

In the limit where we assume $\lambda = -|\lambda|$ constant, $\lambda'_{\text{eff}}(h)$ is always different from zero: local scale invariance is hopelessly lost, and a bounce cannot be found. These expectations are confirmed by our numerical analysis, described in Appendix A.

Actually one finds that a bounce with a SM-like action exists for finite values of x_{max} , but its size R grows to infinity as x_{max} is sent to infinity. Since $h(0)$ is inversely proportional to R , it follows that the bounce is in fact zero everywhere.

However, the absence of a bounce turns out to be an artefact of choosing λ constant in Eq. (68). With λ running there is an extra compensating contribution in Eq. (78), and indeed a bounce of finite size with a SM-like bounce action can be found numerically, provided $M \gg 1/R_{\text{SM}} \approx 10^{17}$ GeV, the scale at which λ has a minimum. Hence we conclude that in such a case the decay probability of the false vacuum is not modified with respect to the SM value.

- II. A second possibility is that λ_6 and λ_8 have opposite signs, but the potential remains bounded from below, hence $\lambda_6 < 0$ and $\lambda_8 > 0$. In this case, the two non-renormalizable terms compensate each other and restore scale invariance locally (in the neighbourhood of field configurations characterized the scale $1/M$). Indeed, $\lambda'_{\text{eff}}(h)$ is zero for

$$h = h(0) = M \sqrt{\frac{2|\lambda_6|}{3\lambda_8}}, \quad (79)$$

and

$$\lambda_{\text{eff}}(h(0)) = -|\lambda| - \frac{2|\lambda_6|^2}{9\lambda_8}. \quad (80)$$

The instanton size is now immediately read off Eq. (76):

$$R = \frac{a}{h(0)\sqrt{|\lambda_{\text{eff}}(h(0))|}} = \frac{a}{M} \frac{1}{\sqrt{\frac{2|\lambda_6|}{3\lambda_8} \left(|\lambda| + \frac{2|\lambda_6|^2}{9\lambda_8} \right)}} \sim \frac{1}{M}. \quad (81)$$

Two important points are to be noted. First, we have in this case

$$|\lambda_{\text{eff}}(h(0))| = |\lambda| + \frac{2|\lambda_6|^2}{9\lambda_8} > |\lambda|, \quad (82)$$

and therefore

$$S[h] < S_{\text{SM}}[h]. \quad (83)$$

As a consequence, the decay probability of the false vacuum can only be increased with respect to the pure SM. Second, the dominant contribution to the tunnelling rate is independent of the scale M of NP, because the action is. This conclusion has been reached in the context of an approximate calculation, but it is easy to see that it holds true in general. Indeed, one may define dimensionless coordinates and field by

$$x = Mr; \quad \tilde{h}(x) = \frac{h(r)}{M}, \quad (84)$$

so that

$$V(h) = M^4 \left[\frac{1}{4} \lambda \tilde{h}^4 + \sum_{n=3}^{\infty} \frac{\lambda_{2n}}{2n} \tilde{h}^{2n} \right] \equiv M^4 \tilde{V}(\tilde{h}). \quad (85)$$

Then

$$\begin{aligned}
S[h] &= 2\pi^2 \int_0^\infty dr r^3 \left[V(h) - \frac{1}{2} h(r) V'(h) \right] \\
&= 2\pi^2 \int_0^\infty dx x^3 \left[\tilde{V}(\tilde{h}) - \frac{1}{2} \tilde{h}(x) \tilde{V}'(\tilde{h}) \right] \quad (86)
\end{aligned}$$

Table 1

Summary of the bounce solutions associated with the potential in Eq. (63). Only the case $M \gg 1/R_{\text{SM}}$ is considered, while for $M \lesssim 1/R_{\text{SM}}$ the bounce action can be any with respect to the SM one.

λ	λ_6	λ_8	h	R	$S[h]$
const	<0	<0	0	∞	0
const	<0	>0	$\neq 0$	$\sim 1/M$	$< S_{\text{SM}}[h]$
const	>0	<0	0	∞	0
const	>0	>0	0	∞	0
run	<0	<0	0	∞	0
run	<0	>0	$\neq 0$	$\sim 1/M$	$< S_{\text{SM}}[h]$
run	>0	<0	0	∞	0
run	>0	>0	$\neq 0$	R_{SM}	$S_{\text{SM}}[h]$

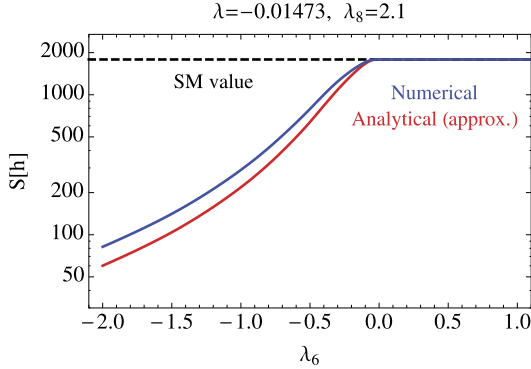


Fig. 4. Numerical vs. analytical (approximated) determination of the bounce action as a function of λ_6 . The relative difference is always within 30%. Formally, for λ constant and $\lambda_6 > 0$ a bounce does not exist; the result in the plot for $\lambda_6 > 0$ has to be understood for λ running.

which is independent of M . This is an important point: whatever modification may the potential in Eq. (64) induce on the tunnelling rate, this is independent of the scale M up to sub-leading pre-exponential corrections.

III. The last case to be considered is the one where the potential is unbounded from below at large field values. When both λ_6 and λ_8 are negative (or at least one is negative and the other is zero), it is clear that the bounce does not exist because of the lost of scale invariance.

The case $\lambda_6 > 0$ and $\lambda_8 < 0$ is more subtle: here a non-zero solution of $\lambda'_{\text{eff}}(h) = 0$ does exist, but it corresponds to a maximum of λ_{eff} and not to a bounce configuration. Actually the fast drop of the potential at large field values makes the whole problem of finding a bounce solution ill-defined in this case: there exist “rolling solutions” that destabilize the electroweak vacuum characterized by a much shorter time scale [16].

These results are summarized in Table 1, with either constant or running λ .⁸ To complete this study, we show in Fig. 4 the comparison between the numerical determination of the bounce action and the analytical approximation in Eq. (75). The agreement is generally quite good. Hence, one can take advantage of the λ_{eff} language in order to describe in an essential way the tunnelling process, as exemplified in Fig. 5, where some potentials corresponding to cases I and II are considered. From that it is clear that modifications of the SM potential above $1/R_{\text{SM}} \approx 10^{17}$ GeV may only shorten the lifetime of the electroweak vacuum.

The above discussion can be generalized to the case of more non-renormalizable terms, including derivative operators, the three

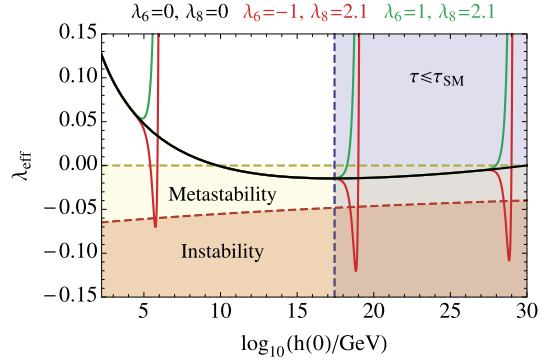


Fig. 5. Evolution of $\lambda_{\text{eff}}(h(0))$ (including the running of λ) for three different values of the NP scale, $M = 10^6$ GeV, M_P and $10^{10}M_P$, and for three different choices of the NP parameters $\lambda_{6,8}$. The instability bound is defined by the inequality $|\lambda_{\text{eff}}(h(0))| > \frac{2\pi^2}{3} \frac{1}{\log(\tau_U/h(0))}$, where $\tau_U = 4.35 \times 10^{17}$ s is the age of the Universe.

main categories being defined by the stability (I and II) or instability (III) of the potential at large field values, and by the presence of a new minimum around the scale M (case II).

Summarizing, the explicit breaking of scale invariance at energy scales above $1/R_{\text{SM}} \approx 10^{17}$ GeV may lead to a new bounce, whose characteristic scale is necessarily related to that of NP (case II). However, this happens only if the NP modifies the ground state of the theory, generating a new deep minimum around the new scale M . In such case, the effect of NP is that of opening a new decay channel for the electroweak vacuum and the tunnelling rate can only increase with respect to the SM calculation. It is then obvious that in this case one is not analyzing anymore the stability of the SM, but that of a different theory, with a completely different ground state.

The above argument also explains why there is an apparent violation of the decoupling theorem, according to which one expects new-degrees of freedom characterized by a scale $M \gg 1/R_{\text{SM}}$ to be irrelevant for the analysis of the vacuum stability. This apparent paradox is due to the fact that the modifications of the effective potential introduced by means of Eq. (63) are not only related to the appearance of high-frequency dynamical modes (for which the decoupling still applies), but they also imply a drastic modification of the ground state of the theory (invalidating the decoupling argument).

We finally note that the modified effective potential of the type in Eq. (63), leading to a fast tunnelling rate (case II), is not a well motivated UV completion of the SM potential close to the Planck scale. On the one hand, except for fine-tuned values of λ_6 and λ_8 , the lifetime of the electroweak vacuum turns out to be extremely fast, in sharp contradiction with the existence of the present Universe. This implies that such modified potential cannot be considered as a phenomenologically viable UV completion of the SM potential. On the other hand, if M is close to the Planck mass, truly gravitational effects [34] cannot be ignored. As recently pointed out in Ref. [35], the latter suppress the tunnelling rate by many orders of magnitude.⁹

4. Conclusions

The decay probability of the electroweak vacuum under quantum tunnelling is intimately related to the breaking of scale in-

⁸ A similar study of the tunnelling rate as a function of λ_6 and λ_8 was performed in Ref. [33].

⁹ Gravitational effects in minimal Einstein gravity tend to slow down the tunnelling rate [34]. However, as long as $1/R_{\text{SM}} \ll M_P$, these corrections are small and hence the presence of gravity does not invalidate the results obtained within the SM in isolation [32].

variance in the Higgs effective potential. The latter can be characterized by an effective h^4 interaction, both within and beyond the SM, which is not scale invariant beyond the tree level. To a good approximation, the size of the bounce and the tunnelling probability are determined by the energy scale at which this effective coupling, $\lambda_{\text{eff}}(h)$, reaches its minimum value and by its value at the minimum, respectively. In the absence of NP, the breaking of scale invariance occurs dominantly via SM radiative corrections. The latter selects a leading bounce characterized by the scale $1/R_{\text{SM}} \approx 10^{17}$ GeV, that implies a sufficiently long lifetime of the electroweak vacuum compared to the age of the Universe.

New degrees of freedom at high energies can, in principle, introduce a new explicit breaking of scale invariance in the effective potential. This, in turn, can shift the energy scale where $\lambda_{\text{eff}}(h)$ reaches its minimum value and, most importantly, can modify such minimum value. If the new degrees of freedom appear above the scale $1/R_{\text{SM}}$, we can distinguish two cases: those where NP simply stabilizes the SM potential, and those where NP introduces new decay channels for the electroweak vacuum. In the first case, characterized by the absence of new minima for $\lambda_{\text{eff}}(h)$, the lifetime of the electroweak vacuum remains unchanged. In the latter case it can only decrease with respect to the pure SM. In particular, a shorter lifetime occurs only if the new degrees of freedom drastically change the ground-state of the theory, introducing a new minimum for $\lambda_{\text{eff}}(h)$, such that it does not make sense anymore to speak about the stability problem of the SM potential.

Given these arguments, we can conclude that the sensitivity of the tunnelling rate to the possible UV completions of the model does not invalidate the vacuum stability analyses performed using the pure SM potential. The scope of the latter is answering the following well-defined physical question: *Does the extrapolation of the SM up to the Planck scale necessarily implies the existence of NP below such scale?* According to the present experimental values of m_h and m_t , we can state that the answer to this question is *no* [1,2]. Were the top mass e.g. 180 GeV, the answer would have been different since NP would have been necessarily implied *well below* the scale where λ reaches its minimum SM value, regardless of the physics in the deep UV which *cannot* improve on stability. We finally stress that the negative answer to the above question does not necessarily imply the absence of NP up to the Planck scale: it only says that the model is compatible with the absence of NP up to the Planck scale. Similarly, from this analysis we cannot infer that any UV completion of the theory at the Planck scale is compatible with the stability of the electroweak vacuum, but only that it is possible to build UV completions that do not contain new degrees of freedom below the Planck scale.

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Appendix A. Numerical determination of the bounce

For the numerical analysis it turns out to be convenient to work with adimensional variables

$$x = Mr; \quad \tilde{h}(x) = \frac{h(r)}{M}; \quad \tilde{V}(\tilde{h}) = \frac{V(h)}{M^4}, \quad (\text{A.1})$$

where M is an arbitrary mass scale. The bounce equation reads

$$\tilde{h}''(x) + \frac{3}{x}\tilde{h}'(x) = \tilde{V}'(\tilde{h}), \quad (\text{A.2})$$

with boundary conditions

$$\lim_{x \rightarrow \infty} \tilde{h}(x) = v/M, \quad (\text{A.3})$$

$$\left. \frac{d\tilde{h}(x)}{dx} \right|_{x=0} = 0. \quad (\text{A.4})$$

A numerical solution can be found by means of the shooting method (see also [8,33,36]), which consists in tuning the initial condition a_0 in such a way that Eq. (A.3) is satisfied at the boundary. In practice, since we cannot start at the singular point $x = 0$, we Taylor-expand the solution around the origin and define the Cauchy problem

$$\tilde{h}(x_{\min}) = a_0 + \frac{1}{8}\tilde{V}'(h(0))x_{\min}^2, \quad (\text{A.5})$$

$$\tilde{h}'(x_{\min}) = \frac{1}{4}\tilde{V}'(h(0))x_{\min}, \quad (\text{A.6})$$

where we kept up to quadratic terms in x_{\min} . By evolving the solution $\tilde{h}(x; a_0)$ up to $x = x_{\max}$, we require

$$\tilde{h}(x_{\max}; \bar{a}_0) = 0, \quad (\text{A.7})$$

which determines the value of \bar{a}_0 which satisfies the boundary condition at $x = x_{\max}$.

Analogously, by Taylor-expanding the solution at the first non-trivial order in $1/x_{\max}$ for large x_{\max} , we get ($m = 0$ case)

$$\tilde{h}(x_{\max}) = \frac{a_{\infty}}{x_{\max}^2}, \quad (\text{A.8})$$

which is useful in order to have an analytical control over the asymptotic solution. On the other hand, if the mass term is kept in the potential ($m \neq 0$ case) the asymptotic solution is (see e.g. Appendix of [37])

$$\tilde{h}(x_{\max}) = a_{\infty} \frac{\epsilon}{x_{\max}} K_1(\epsilon x_{\max}) \approx a_{\infty} \frac{\epsilon}{x_{\max}} \sqrt{\frac{\pi}{\epsilon x_{\max}}} e^{-\epsilon x_{\max}}, \quad (\text{A.9})$$

where $\epsilon = m/M$ and in the last step we used the asymptotic expression of the modified Bessel function $K_1(x)$ for large x .

The bounce action is determined by integrating the profile of the bounce $h(x; \bar{a}_0)$ between x_{\min} and x_{\max}

$$S[h] = \frac{2\pi^2}{M^4} \int_{x_{\min}}^{x_{\max}} x^3 dx \left[V(\tilde{h}M) - \frac{1}{2}\tilde{h}M V'(\tilde{h}M) \right], \quad (\text{A.10})$$

where we applied the equation of motion and performed the change of variables in Eq. (A.1). Finally, the algorithm is iterated by choosing increasingly smaller (larger) values of x_{\min} (x_{\max}).

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